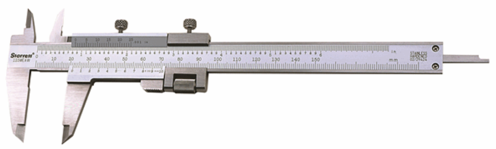
Problem Set 5

Note: The problem sets serve as additional exercise problems for your own practice. Problem Set 5 covers materials from §4.2 – §4.5.

1. Estimate the value of each of the following, using an appropriate linear approximation but without using a calculator.
   (a) $\ln 0.98$
   (b) 1.013^{10}
   (c) $f(3.02)$, where $f(x) = \dfrac{2x}{1+x^2}$
2. Find the linear approximation of each of the following functions at the given real number.
   (a) $f(x) = (1+x)^p$ at $x = 0$, where p is a fixed real number
   (b) $g(\theta) = \sec\theta$ at $\theta = \frac{\pi}{3}$
   (c) $h(t) = \ln\ln t$ at $t = e$
3. The diameter of a sphere is measured using a Vernier caliper, and the maximum percentage error of this measurement is estimated to be 0.1%. If this measurement is then used in calculating the volume of the sphere, what will be the maximum percentage error of the calculated volume?



Vernier caliper

4. Find the first iteration, correct to 4 decimal places, in solving the equation
$$x = 1 + 0.5\sin x$$
using Newton's method with initial guess $x_0 = 1.5$.
5. Consider the equation
$$x^3 + 2x = 4.$$
   (a) Show that the equation has exactly one solution on $[1, 2]$.
   (b) Using Newton's method, find the solution to the equation correct to 4 decimal places.
6. Approximate the value of e^π to 4 decimal places by solving the equation
$$\sin\ln x = 0$$
numerically using Newton's method. Since we know that $e \approx 3$ and $\pi \approx 3$, let's use the initial trial $x_0 = 3^3 = 27$. Does it make a difference if we had used other initial trials x_0?

7. Let $f: \mathbb{R} \to \mathbb{R}$ be a function defined by

$$f(x) = \begin{cases} xe^{-1/x^2} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}.$$

(a) Show that f is continuous at 0.

(b) Show that f is differentiable at 0.

(c) Is f' continuous at 0?

8. Recall that in Q4 of Problem Set 4, we studied the continuous function $f: \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} \dfrac{\sin 2x + \sin x}{e^x - 1} & \text{if } x < 0 \\ 3e^x - x & \text{if } x \geq 0 \end{cases}.$$

Now determine whether f is differentiable at 0 or not.

9. Evaluate each of the following limits, if it exists. You may use l'Hôpital's rule or any other method learnt in this course whenever appropriate.

(a) $\displaystyle\lim_{x\to 1} \frac{x - e^{x-1}}{(x-1)^2}$

(b) $\displaystyle\lim_{x\to 0^+} \frac{\cot x}{\csc x}$

(c) $\displaystyle\lim_{x\to 0^+} (\sin x)^x$

(d) $\displaystyle\lim_{x\to 0} \frac{x^2 \sin\frac{1}{x}}{\sin x}$

(e) $\displaystyle\lim_{x\to e^+} (\ln x)^{\frac{1}{x-e}}$

(f) $\displaystyle\lim_{x\to 0} \frac{x \tan x}{1 - \sqrt{1 - x^2}}$

(g) $\displaystyle\lim_{x\to 1} \frac{1 + \ln x - x^x}{1 + \ln x - x}$

(h) $\displaystyle\lim_{x\to 0} \left(\frac{\sin x}{x}\right)^{\frac{1}{x^2}}$

10. Let a be a real number and f be a function whose second derivative f'' is continuous at a. Show that

$$\lim_{x\to a} \frac{f(x) - \left(f(a) + f'(a)(x-a) + \dfrac{f''(a)}{2!}(x-a)^2\right)}{(x-a)^2} = 0.$$

11. Find the coordinates of all the global maximum or minimum points of each of the following functions on the given interval.

(a) $f(x) = \sqrt{3}\sin x + \cos x$ on $[0, \pi]$

(b) $g(x) = \dfrac{x^2}{x-1}$ on $[-1, 3]$

(c) $h(x) = |x-3| + |x+2|$ on $[-4, 4]$